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SOME PROBABILITY MODELS RELATED TO CLASSICAL OCCUPANCY WITH POSSIBLE APPLICATIONS TO TACTICAL ANALYSIS

by

Bruno O. Shubert

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In this report three types of probability distributions associated with random search-type models are derived. The distributions are closely related to the classical occupancy problem. The first two types are univariate and are applicable to simple random search models such as mine hunting or one-kind mine sweeping. The third type is multivariate allowing for various mine ship count settings.

Some Probability Models Related

To Classical Occupancy With Possible

Applications To Tactical Analysis

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Abstract

In this report three types of probability distributions associated with random search-type models are derived. The distributions are closely related to the classical occupancy problem. The first two types are univariate and are applicable to simple random search models such as mine hunting or one-kind mine sweeping. The third type is multivariate allowing for various mine ship count settings.

1. Introduction

In this report we investigate some discrete probability models which may be useful in search and related situations. The models are closely related to the classical occupancy problem of probability theory and may in fact be viewed as generalizations of it.

The models are discrete in the sense that there are m discrete locations - referred to as cells - possibly containing the objects of the search - referred to as balls. The search itself is random in the sense that the cell to be searched is selected at random, i.e. with equal probability among all the cells involved.

Two univariate models and a multivariate extension of the first of the two are considered. The detailed description of the first univariate model - called here Random Search-Variant 1 - is as follows:

There are m cells, y_0 of them initially empty and y_1 of them containing a single ball each. The search consist of repeated independent trials, where, at each trial, a cell is selected with probability $1/m$. The selected cell is examined, and if it contains a ball the ball is found with probability p. If a ball is found, it is removed from the cell so that the cell is now empty. Since the trials leading to a cell selection are assumed independent the same cell can be selected again regardless of the results of previous trials. Thus a cell can be selected even if a ball was found and removed from it at some previous trial.

This model is investigated in Section 3. We find the probability distribution of the number of balls found in n trials, the number of trials needed to find the k-th ball as well as moments of these distributions.

As mentioned earlier, this model can be rephrased as an occupancy problem of distributing n balls into m cells. In fact, if $y_0 = 0$ and

$p = 1$ it is identical with the classical occupancy problem - merely identify the number of balls found in our model with the final number of nonempty cells in the occupancy problem.

The second univariate model - Random Search Variant 2 - differs from Variant 1 only in the rule for selecting cells at each trial. This time we assume that the cell in which a ball has been found and removed is no longer a candidate for further selections. Thus, the number of cells to be searched is reduced by one each time a ball is found. The corresponding distributions of the number of balls found, the number of trials needed to find the k-th ball and their moments are described in Section 4. It may be interesting to notice that, with $p = 1$, this model is a hybrid of a classical sampling with and without replacement. Think of the m cells as m balls, y_0 of them white and y_1 black. Then draw n balls replacing the white ones but not replacing the black ones, and ask about the distribution of the number of black balls drawn.

The last section of this report deals with a multivariate extension of the Variant 1 model. The extension consists in assuming that there are now s different types of balls involved and each time a ball of type $j > 0$ is found it is replaced by a ball of type $j - 1$ or removed if $j - 1 = 0$. The model parameters are now nonnegative integers $y_0, \ldots, y_s$, where $y_0 + \ldots + y_s = m$, y_0 is the number of empty cells and y_j is the number of cells with balls of type j each initially. For simplicity, only the value $p = 1$ is considered here, i.e. if a cell containing a ball is selected the ball is found (and its type reduced) with probability one.

The cell selection process is the same as in Variant 1. That is, cells are selected at random among the m cells in independent trials so that the same cell can be selected repeatedly regardless of the ball type it contains.

Again, this model can be rephrased as an occupancy problem of distributing n balls into m cells and interpreting a cell with ball type j as a cell containing $s - j$ balls for $j > 0$ or a cell containing at least s balls for $j = 0$. However, classical occupancy problems invariably assume that that initially all cells are empty, i.e. that $y_s = m$ in our notation. Therefore, classical results are not directly applicable to our model.

In Section 5 we find the first two (joint) moments of the resulting numbers of balls type $j = 0,\dots,s$ after n trials as well as of the numbers of each ball type found. An expression for the marginal distribution of the number of balls type j is also found - although the complexity of this expression leaves some doubts about its possible uses.

In this report, no attempt is made to study various asymptotic distribution resulting in letting the parameters of the models increase to infinity in some way. Although this may yield a considerable simplification, for instance by a Poisson distribution in the case of Variant 1, in most of the applications intended here the values of the parameters are typically small. Still, asymptotic results may perhaps warrant future investigation.

Likewise, no algorithms to actually compute numerical values for the derived quantities are presented in this report - mainly because of a limited time and each of adequate computational facilities during preparation of this report. This, again, is left for a possible future work.

2. Possible Applications

Although the purpose of this report is to develop model building tools rather then real world models themselves, a few possibilities will be mentioned briefly.

A rather straightforward application of the two univariate cases would apply when one indeed has m distinct geographical locations out of which y_1 contain at most one target each. Searching these locations at random would imply an absence of any particular search strategy or any further information about the locations of these targets. Such a model may, for instance, serve as a basis against which an efficiency of various search strategies can be compared. In the case when the targets are distributed in a continuous geographical region, the discrete cells may be defined artificially by partitioning the region. The actual size of a cell must be chosen to satisfy the conflicting requirements of the size of the searcher's detection region and the assumption of a single target in each cell. This, of course, is the problem encountered any time a continuous model is being discretized. The choice between Variants 1 and 2 is then dictated by the nature of the targets and/or searcher. For example, if the targets are stationary, Variant 2 is more appropriate. Typical example here could be underwater mine hunting. If the targets are mobile and can randomly rearrange between each search trial, Variant 1 could be used. Another possibility leading to Variant 1 is when the searcher (in the general sense of this term) has no control over the cell selection. Typical application in this case is of course an artillery coverage model - delivering n shells to a region partitioned into m cells out of which y_1 contain a target. Similar applications, not involving an actual physical search, are conceivable. For instance, in reliability one can have an equipment with two kinds of components - one with potentially unlimited supply of spares and the other with any a small number y_1 of spares.

The possible applications of the multivariate version are even more versatile depending on the interpretation of the ball type. Consider, for instance, a mine sweeping model with a ball type being the ship count the mine (ball) is initially set at. Alternately, consider a mine field with cells corresponding to paths through the mine field and ball type corresponding to the number of mines along a path. The cells would be separated from each other by the actuation width of a mine in this application. Yet another application is again the artillery coverage with multiple hits required to destroy a target.

Our final remark concerns the choice of the model parameters. They can, of course, be left as true parameters to investigate their influence on the quantity of interest. But they themselves can be considered random variables resulting in various mixtures of the probability distributions involved. For instance, the number y_1 of targets could be a Binomial random variable thus modeling the effect of a random appearance of targets. Even more realistically, perhaps, the ship counts in the mine sweeping application can be considered set at random among some given range of ship counts. Such assumptions could sometimes even simplify the analysis as shown e.g. at the end of Section 5.

3. Random Search - Variant 1

For the sake of reference let us restate the assumption of the model. We have m cells, y_0 of them empty and y_1 of them containing a single ball each. Cells are being selected repeatedly with equal probability $1/m$ in independent trials. If a selected cell contains a ball the ball is removed with probability $p > 0$ and remains there with probability $q = 1 - p$ independently of the results of previous trials. Thus, in Variant 1, the same cell can be selected again regardless of the event that a ball may have been removed from it before.

The parameters of the model are

$$y_0 \, , \, y_1 \, , \text{ and } p \, ,$$

where y_0 , y_1 are nonnegative integers and $0 < p \leq 1$. The letter $m = y_0 + y_1$.

Let $K(n)$ be the number of balls obtained at the conclusion of the n-th trial. Clearly $0 \leq K(n) \leq \min\{n, y_1\}$ and

$$(3.1) \qquad K(n+1) = \begin{cases} K(n) & \text{with probability } 1 - \dfrac{y_1 - K(n)}{m} p , \\ \\ K(n) + 1 & \text{with probability } \dfrac{y_1 - K(n)}{m} p , \end{cases}$$

for $n = 0,1,\ldots$ with $K(o) = 0$.

Hence calling

$$P_n(k) = P(K(n) = k) ,$$

we have the recurrence

$$(3.2) \qquad \begin{aligned} P_{n+1}(k) &= \left(1 - \frac{y_1 - k}{m} p\right) P_n(k) \\ &+ \frac{y_1 - k + 1}{m} p \, P_n(k-1) , \end{aligned}$$

$n = 0,1,\ldots$; with $P_n(-1) = 0$ and $P_o(0) = 1$, $P_o(k) = 0$ for $k > 0$.

This can be used to recursively evaluate the distributions $P_n(k)$ for given values of the parameters m, y_1 and p . However, it is also possible to obtain an explicit formula, for instance by employing the generating functions

$$(3.3) \qquad G_k(t) = \sum_{n=0}^{\infty} t^n P_n(k) \quad , \; k = 0,1,\ldots,y_1 .$$

From (3.2) with $k = 0$ we have

$$P_{n+1}(0) = \left(1 - \frac{y_1}{m} p\right) P_n(0) ,$$

and using $P_0(0) = 1$

$$P_n(0) = \left(1 - \frac{y_1}{m} p\right)^n \quad , n = 0,1,\ldots .$$

Hence

(3.4) $$G_0(t) = \frac{1}{1 - t\left(1 - \frac{y_1}{m} p\right)} .$$

From (3.2) with $k > 0$ by multiplying by t^{n+1} and summing over $n = 0,1,\ldots$ we obtain

$$G_k(t) = t\left(1 - \frac{y_1 - k}{m} p\right) G_k(t) + t \frac{y_1 - k + 1}{m} p \, G_{k-1}(t) ,$$

whence solving for $G_k(t)$ and iterating

(3.5) $$G_k(t) = \frac{y_1^{(k)} \left(\frac{pt}{m}\right)^k}{\prod_{j=0}^{k} \left[1 - t\left(1 - \frac{y_1 - j}{m} p\right)\right]} \quad , k = 0,\ldots,y_1 ,$$

where $y_1^{(k)} = \frac{y_1!}{(y_1 - k)!}$. Since the $k + 1$ roots $t_j = a_j^{-1}$, $a_j = 1 - \frac{y_1 - j}{m} p$, of the denominator are all distinct we can use the partial fraction expansion

$$\left(\prod_{i=0}^{k} (1 - ta_i)\right)^{-1} = \sum_{j=0}^{k} A_j(1 - ta_j)^{-1} ,$$

where $$A_j = a_j^k \left(\prod_{\substack{i=0\\ i\neq j}}^{k} (a_j - a_i)\right)^{-1} , \; j = 0,\dots, k .$$

With $a_j = 1 - \dfrac{y_1 - j}{m} p$ the coefficients are

$$A_j = \left(1 - \frac{y_1 - j}{m} p\right) \left(\frac{m}{p}\right)^k \left(\prod_{\substack{i=0\\ i\neq j}}^{k} (j-i)\right)^{-1}$$

$$= \left(1 - \frac{y_1 - j}{m} p\right)^k \left(\frac{m}{p}\right)^k \frac{(-1)^{k-j}}{j!(k-j)!} .$$

Upon substitution into (3.5) we have

$$G_k(t) = y_1^{(k)} t^k \sum_{j=0}^{k} (-1)^{k-j} \frac{\left(1 - \frac{y_1 - j}{m} p\right)^k}{j!(k-j)!} (1 - ta_j)^{-1} ,$$

whence expanding $(1 - ta_j)^{-1}$ into a geometric series

$$G_k(t) = \sum_{\nu=0}^{\infty} t^{\nu+k} \binom{y_1}{k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \left(1 - \frac{y_1 - j}{m} p\right)^{\nu+k} .$$

The coefficients of this expansion are the desired probabilities, i.e.

(3.6)
$$P_n(k) = \binom{y_1}{k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \left(1 - \frac{y_1 - j}{m} p\right)^n$$

$$= \left(\frac{p}{m}\right)^n \binom{y_1}{k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left(\frac{m}{p} - y_1 + k - j\right)^n ,$$

$$k = 0,1,\ldots,y_1 \; ; \; n = 0,1,\ldots \; .$$

Remark:

(3.7) Since $\sum_{j=0}^{k} (-1)^j \binom{k}{j} (x + k - j)^n = \Delta^k x^n$

with Δ the forward difference operator one can also write

$$P_n(k) = \left(\frac{p}{m}\right)^n \binom{y_1}{k} \Delta^k \left(\frac{m}{p} - y_1\right)^n .$$

With $y_1 = m$ and $p = 1$ this is the classical occupancy distribution ([Fl], p. 58) although the generalizations with $y_1 < m$ or $p < 1$ can also be found in the literature ([JK], p. 124, 140). Note that the formula (3.6) gives automatically $P_n(k) = 0$ for $k > n$.

Next, let us look at the moments of this distribution. The expectation $\mu_n = E[K(n)]$ (and higher moments as well) can be derived from the recurrence (3.2) by conditioning. For instance taking conditional expectation we obtain

$$\mu_{n+1} = (1 - \frac{p}{m})\mu_n + y_1 \frac{p}{m} , \qquad n = 0,1,\ldots ,$$

$$\mu_0 = 0 \text{ , whence}$$

$$\mu_n = y_1\left[1 - (1 - \frac{p}{m})^n\right] , \qquad n = 0,1,\ldots .$$

However, one can obtain all factorial moments

$$\mu_n^{(r)} = E[K(n)(K(n) - 1)\ldots(K(n) - r + 1)] ,$$

$r = 1,\ldots,y_1$,

directly from (3.7) since

$$\mu_n^{(r)} = \sum_{k=r}^{y_1} k^{(r)} P_n(k) = (\frac{p}{m})^n \sum_{k=r}^{y_1} \frac{y_1!}{(y_1-k)!(k-r)!} \Delta^k(\frac{m}{p} - y_1)^n$$

$$= (\frac{p}{m})^n y_1^{(r)} \sum_{j=0}^{y_1-r} (y_1 - r)^{(j)} \frac{\Delta^{j+r}(\frac{m}{p} - y_1)^n}{j!}$$

$$= y_1^{(r)}(\frac{p}{m})^n \Delta^r(\frac{m}{p} - r)^n ,$$

by using the identity

$$\sum_{j=0}^{\nu} \nu^{(j)} \frac{\Delta^j \chi^n}{j!} = (\chi + \nu)^n$$

and linearity of the difference operator. Equivalently

$$\mu_n^{(r)} = y_1^{(r)} \sum_{j=0}^{r} (-1)^j \binom{r}{j} \left(1 - \frac{p}{m} j\right)^n ,$$

$$r = 1,\ldots,y_1 ,$$

whence again the expectation

$$\mu_n^{(1)} = y_1 [1 - (1 - \frac{p}{m})^n] .$$

From $\mu_n^{(2)} = y_1^{(2)} \left[1 - 2(1 - \frac{p}{m})^n + (1 - \frac{2p}{m})^n\right]$ we can get the variance

$\sigma_n^2 = \mathrm{Var}[K(n)]$,

$$\sigma_n^2 = y_1 (1 - \frac{p}{m})^n \left[1 - y_1 (1 - \frac{p}{m})^n\right]$$

$$- y_1^{(2)} (1 - \frac{2p}{m})^n , \qquad n = 0,1,\ldots ,$$

where $y_1^{(2)} = y_1(y_1 - 1)$.

Waiting time distributions:

Consider next the random variable W_k defined to be the trial number at which the k-th ball is found. We have

$$W_k = T_1 + \ldots + T_k , \qquad k = 1,\ldots,y_1 ,$$

where T_j is the number of trials to find the j-th ball counted from the the trial at which the (j-1)-th ball was found. Clearly,

T_j is geometric with the parameter $\frac{y_1 - j + 1}{m} p$ and $T_1, T_2, \ldots$ are independent. Thus calling

$$\phi_k(t) = \sum_{n=0}^{\infty} t^n P(W_k = n)$$

the probability generating function of W_k we have immediately

$$\phi_k(t) = \prod_{j=1}^{k} \frac{t \frac{y_1 - j + 1}{m} p}{1 - t(1 - \frac{y_1 - j + 1}{m} p)}$$

$$= y_1^{(k)} (\frac{pt}{m})^k \prod_{j=0}^{k-1} \left[1 - t(1 - \frac{y_1 - j}{m} p)\right]^{-1}$$

$$= (y_1 - k + 1) \frac{pt}{m} G_{k-1}(t) ,$$

where G is the generating function (3.3).

Hence

$$\phi_k(t) = \sum_{\nu=0}^{\infty} t^{\nu+k}(y_1 - k + 1)\,\frac{p}{m}$$

$$\binom{y_1}{k-1} \sum_{j=0}^{k-1} (-1)^{k-1-j}\binom{k-1}{j}\left(1 - \frac{y_1 - j}{m}\,p\right)^{\nu+k-1} ,$$

and thus

$$P(W_k = n) = \frac{pk}{m}\binom{y_1}{k} \sum_{j=0}^{k-1} (-1)^{k-1-j}\binom{k-1}{j}\left(1 - \frac{y_1 - j}{m}\,p\right)^{n-1}$$

$$= \left(\frac{p}{m}\right)^n k\binom{y_1}{k} \sum_{j=0}^{k-1} (-1)^{j}\binom{k-1}{j}\left(\frac{m}{p} - y_1 + k - 1 - j\right)^{n-1} ,$$

$$n = k,\ k+1, \ldots\ ;\quad k = 1, \ldots, y_1 .$$

Again, this can be reexpressed as

$$P(W_k = n) = \left(\frac{p}{m}\right)^n k\binom{y_1}{k}\,\Delta^{k-1}\left(\frac{m}{p} - y_1\right)^{n-1} .$$

The expectation and the variance can be gotten most easily by using the fact that W_k is a sum of independent geometric random variables T_j with

$$E[T_j] = \frac{m}{p(y_1-j+1)} ,\quad Var[T_j] = \frac{m^2}{p^2(y_1-j+1)^2}\left[1 - \frac{y_1-j+1}{m}\,p\right]$$

so that

$$E[W_k] = \frac{m}{p} \sum_{j=0}^{k-1} \frac{1}{y_1 - j} ,$$

and

$$Var[W_k] = \frac{m}{p} \sum_{j=0}^{k-1} \frac{\frac{m}{p} - y_1 + j}{(y_1 - j)^2} .$$

4. Random Search-Variant 2

This model differs from Variant 1 only in that the cell from which a ball has been removed is not selected again. The development parallels that of Variant 1 very closely and the same notation and definitions as in Section 3 are used.

The basic recurrence for the random variables $K(n)$, $n = 0,1,\ldots,$ is now

$$K(n+1) = \begin{cases} K(n) & \text{with probability } 1 - \dfrac{y_1 - K(n)}{m - K(n)}\, p \,, \\[2ex] K(n) + 1 & \text{with probability } \dfrac{y_1 - K(n)}{m - K(n)}\, p \,, \end{cases}$$

with $K(0) = 0$. To avoid the trivial case $y_0 = 0$ we assume that $y_0 > 0$, i.e. $y_1 < m$.

The recurrence for the distribution $P_n(k)$ is now

$$P_{n+1}(k) = \left(1 - \frac{y_1 - k}{m - k}\, p\right) P_n(k) + \frac{y_1 - k + 1}{m - k + 1}\, p\, P_n(k-1) \,,$$

whence the generating functions $G_k(t)$ are

$$G_k(t) = \frac{\dfrac{y_1^{(k)}}{m^{(k)}}\,(pt)^k}{\prod\limits_{j=0}^{k}\left[1 - t\left(1 - \dfrac{y_1 - j}{m - j}\, p\right)\right]} \,,$$

and calling

$$a_j = 1 - \frac{y_1 - j}{m - j} p ,$$

we have

$$a_j - a_i = \frac{y_0 p(j-i)}{(m-i)(m-j)} .$$

Thus

$$A_j = \left(1 - \frac{y_1 - j}{m - j} p\right)^k \frac{m - j}{(y_0 p)^k} \prod_{\substack{i=0 \\ i \neq j}}^{k} \frac{m - i}{j - i}$$

$$= \left(1 - \frac{y_1 - j}{m - j} p\right)^k \frac{m^{(k)}}{(y_0 p)^k} \frac{(-1)^{k-j}}{j!(k-j)!} ,$$

and hence

$$G_k(t) = \sum_{\nu=0}^{\infty} t^{\nu+k} \frac{1}{y_0^k} \binom{y_1}{k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \left(1 - \frac{y_1 - j}{m - j} p\right)^{\nu+k} .$$

From here the probabilities are

$$P_n(k) = \frac{1}{y_0^k} \binom{y_1}{k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \left(1 - \frac{y_1 - j}{m - j} p\right)^n ,$$

$k = 0,\ldots,y_1$, $\quad n = 0,1,\ldots$.

Since with $q = 1 - p$

$$1 - \frac{y_1 - j}{m - j} p = q + \frac{y_0 p}{m - j} ,$$

we can also write

$$P_n(k) = \frac{1}{y_0^k}\binom{y_1}{k} \sum_{j=0}^{k} (-1)^j \binom{k}{j}\left(q + \frac{y_0 p}{m - k + j}\right)^n$$

$$= \frac{1}{y_0^k}\binom{y_1}{k} \Delta^k F^n(0)$$

by using the identity

$$\Delta^k F(x) = \sum_{j=0}^{k} (-1)^k \binom{k}{j} F(x + k - j)$$

with $$F(x) = q + \frac{y_0 p}{m - x} .$$

The last expression can be used to obtain the factorial moments although the results are somewhat more complex that for Variant 1. We have for $r - 0,\ldots,y_1$

$$\mu_n^{(r)} = \sum_{k=r}^{y_1} k^{(r)} P_n(k)$$

$$= \frac{y_1^{(r)}}{y_0^{n+r}} \sum_{j=0}^{y_1-r} \binom{y_1-r}{j} y_0^{n-j} \Delta^j(\Delta^r F^n(0))$$

$$= \frac{y_1^{(r)}}{y_0^{n+r}} \sum_{j=0}^{y_1-r} \binom{y_1-r}{j} (y_0-1)^j \Delta^r F^n(n-j)$$

by using the identity

$$\sum_{j=0}^{b} \binom{b}{j} \alpha^{b-j} \Delta^j g(x) = \sum_{j=0}^{b} \binom{n}{j} (\alpha-1)^j g(x+b-j)$$

with $b = y_1 - r$, $\alpha = y_0$ and $g(x) = \Delta^r F^n(x)$.

Removing the difference operator and substituting for $F(x)$ we obtain the formula

$$\mu_n^{(r)} = \frac{y_1^{(r)}}{y_0^{n+r}} \sum_{i=0}^{r} \sum_{j=0}^{y_1-r} (-1)^i \binom{r}{i} \binom{y_1-r}{j} (y_0-1)^j$$

$$\left(q + \frac{y_0 p}{m+i+j-n-r} \right)^n , \qquad r = 0,\ldots,y_1 .$$

In particular, the expectation is $(r = 1)$

$$E[K(n)] = \frac{y_1}{y_0^{n+1}} \sum_{j=0}^{y_1-1} \binom{y_1-1}{j} (y_0-1)^j$$

$$\left[\left(q + \frac{y_0 p}{m-n+j-1} \right)^n - \left(q + \frac{y_0 p}{m-n+j} \right) \right] .$$

The variance $Var[K(n)] = \mu_n^{(2)} + E[K(n)] - E^2[K(n)]$ can be obtained similarly, resulting in a rather long expression.

Waiting time distribution - Variant 2

The derivation is analogous to that of Variant 1 with the only difference being the parameter $\frac{y_1 - j + 1}{m - j + 1}\,p$ of the geometric random variables T_j .

The probability generating function $\phi_k(t)$ of the waiting time W_k is now

$$\phi_k(t) = \frac{y_1^{(k)}}{m^{(k)}} (pt)^k \prod_{j=0}^{k-1} \left[1 - t\left(1 - \frac{y_1 - j}{m - j}\,p\right)\right]^{-1}$$

$$= \frac{y_1 - k + 1}{m - k + 1}\, pt\, G_{k-1}(t)$$

$$= \sum_{\nu=0}^{\infty} t^{\nu+k} \frac{pk}{y_0^k (m-k+1)} \binom{y_1}{k} \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j}$$

$$\left(1 - \frac{y_1 - j}{m - j}\,p\right)^{\nu+k-1} ,$$

so that

$$P(W_k = n) = \frac{pk}{y_0^k (m-k+1)} \binom{y_1}{k} \sum_{j=0}^{k-1} (-1)^j$$

$$\binom{k-1}{j} \left(q + \frac{y_0 p}{m-k+1+j}\right)^{n-1} ,$$

$n = k, k+1, \ldots$, $k = 1, \ldots, y_1$.

From

$$E[T_j] = \frac{m - j + 1}{p(y_1-j+1)} ,$$

$$Var[T_j] = \frac{(m-j+1)^2}{p^2(y_1-j+1)^2} \left[1 - \frac{y_1 - j + 1}{m - j + 1} p\right] ,$$

we get immediately

$$E[W_k] = \frac{1}{p} \sum_{j=0}^{k-1} \frac{m - j}{y_1 - j} = \frac{k}{p} + \frac{y_0}{p} \sum_{j=0}^{k-1} \frac{1}{y_1 - j}$$

and

$$Var[W_k] = \frac{1}{p^2} \sum_{j=0}^{k-1} \left(\frac{m-j}{y_1-j}\right)^2 \left(1 - \frac{y_1-j}{m-j} p\right)$$

$$= \frac{kq}{p^2} + \frac{y_0}{p^2} \sum_{j=0}^{k-1} \frac{(1+q)(y_1-j) + y_0}{(y_1-j)^2} .$$

5. A Multivariate Search Model

In this section we investigate some aspects of the multivariate generalization of the Variant 1 model. We have again m cells, y_0 of them empty, and $m-y_0$ of them containing one ball each. The balls can be of $s \geq 1$ different types and initially, we have y_1 cells containing balls of type 1, y_2 cells containing ball type 2, etc. We shall refer to the empty cells as containing ball type 0, i.e. no ball. Thus the initial configuration is specified by a vector of nonnegative integers

$$\underline{y} = (y_0, \ldots, y_s)^T ,$$

where $y_0 + \ldots + y_s = m$ and y_j is the number of cells containing a ball of type j each.

Cells are searched in repeated trials where at each trial a cell is selected at random equally likely among the m cells and independently of previous trials. If the selected cell contains a ball of type $j > 0$ the ball is replaced by a ball of type $j - 1$. Balls type 0 are just replaced, i.e. empty cells remain empty. The same cell can be selected repeatedly regardless of the result of previous trials. Thus for $s = 1$ this is equivalent to the Variant 1 model with $p = 1$. The model is therefore specified by a single vector parameter $\underline{y}$.

Let $\underline{X}(n) = (X_0(n), \ldots, X_s(n))^T$, $n = 0,1,\ldots,$ be random vector where for each $j = 0,\ldots, s$

$$X_j(n) = \text{number of cells of type } j \text{ after the } n^{\underline{th}} \text{ trial.}$$

Then $\underline{X}(0) = \underline{y}$ and for $n = 0,1,\ldots$

$$(5.1)\qquad X_0(n+1) = \begin{cases} X_0(n) & \text{with probability } 1 - \dfrac{X_1(n)}{m} , \\ \\ X_0(n) + 1 & \text{with probability } \dfrac{X_1(n)}{m} , \end{cases}$$

$$X_j(n+1) \begin{cases} X_j(n) - 1 & \text{with probability } \dfrac{X_j(n)}{m} , \\ X_j(n) + 1 & \text{with probability } \dfrac{X_{j+1}(n)}{m} , \\ X_j(n) & \text{with probability } 1 - \dfrac{X_j(n) + X_{j+1}(n)}{m} , \end{cases}$$

for $0 < j < s$, and

$$X_s(n+1) = \begin{cases} X_s(n) - 1 & \text{with probability } \dfrac{X_s(n)}{m} , \\ \\ X_s(n) & \text{with probability } 1 - \dfrac{X_s(n)}{m} . \end{cases}$$

Of course

$$(5.2)\qquad X_0(n) + X_1(n) + \ldots + X_s(n) = m$$

for all $n = 0,1,\ldots$.

Alternately, let

$$\underline{K}(n) = (K_0(n),\ldots, K_s(n))^T$$

be a random vector where for $j = 0,\ldots, s$.

$K_j(n)$ = number of trials (up to n-th) at which ball type j was found.

Then $\underline{K}(0) = \underline{0}$ (zero vector) and

$$K_0(n+1) = \begin{cases} K_0(n) + 1 & \text{with probability} \quad \dfrac{y_0 + K_1(n)}{m}, \\ K_0(n) & \text{with probability} \quad 1 - \dfrac{y_0 + K_1(n)}{m}, \end{cases}$$

$$K_j(n+1) = \begin{cases} K_j(n) + 1 & \text{with probability} \quad \dfrac{y_j + K_{j+1}(n) - K_j(n)}{m}, \\ K_j(n) & \text{with probability} \quad 1 - \dfrac{y_j + K_{j+1}(n) - K_j(n)}{m}, \end{cases}$$

for $0 < j < s$, and

$$K_s(n+1) = \begin{cases} K_s(n) + 1 & \text{with probability} \quad \dfrac{y_s - K_s(n)}{m}, \\ K_s(n) & \text{with probability} \quad 1 - \dfrac{y_s - K_s(n)}{m}, \end{cases}$$

This time for all $n = 0,1,\ldots$

$$K_0(n) + \ldots + K_s(n) = n .$$

There is a simple linear relation between $\underline{X}(n)$ and $\underline{K}(n)$, namely

$$X_0(n) = y_0 + K_1(n) \ ,$$

(5.3) $$X_j(n) = y_j - K_j(n) + K_{j+1}(n) \ , \quad 0 < j < s \ ,$$

$$X_s(n) = y_s - K_s(n) \ ,$$

and conversely for $0 < j \leq s$

$$K_j(n) = y_j + \ldots + y_s - (X_j(n) + \ldots + X_s(n)) \ ,$$

with

$$K_0(n) = n - (K_1(n) + \ldots + K_s(n)) \ .$$

Note that in terms of the mine sweeping model $K_1(n)$ is the number of deactivated (exploded) mines. Also $B(n) = K_1(n) + \ldots + K_s(n)$, the total number of balls found, is the total number of contacts the sweeper makes during n sweeps.

Either of the recurrence relations yields immediately the corresponding recurrence for the joint probability distribution of the random vector involved. For instance denoting

$$P_n(k_1,\ldots,k_s) = P(K_1(n) = k_1,\ldots,K_s(n) = k_s)$$

we have for $n = 0,1,\ldots$.

$$P_{n+1}(k_1,\ldots,k_s) = \frac{y_0 + k_1}{m} P_n(k_1,\ldots,k_s)$$

$$+ \sum_{j=1}^{s-1} \frac{y_j + k_{j+1} - k_j + 1}{m} P_n(k_1,\ldots,k_{j-1},\ k_j-1,\ k_j,\ldots,k_s)$$

$$+ \frac{y_s - k_s + 1}{m} P_n(k_1,\ldots,k_{s-1},\ k_s-1)$$

with initial condition $P_0(k_1,\ldots,k_s) = 1$ if $k_1 = \ldots = k_s = 0$, and boundary conditions $P_n(k_1,\ldots,k_s) = 0$ whenever $k \ < 0$ for some $j = 1,\ldots,s$.

In principle, this can be used to recursively evaluate the distribution $P_n(k_1,\ldots,k_s)$ but for larger values of s this is not practical since the storage requirements increase exponentially with s . Generally, an array of dimension $(y_s + 1)(y_s + y_{s-1} + 1) \ldots (y_s + \ldots + y_1 + 1)$ would be needed to store the current values of P_n although some savings can still be made due to the fact that $P_n = 0$ unless
$0 \leq k_j \leq y_j + k_{j+1}$, $1 \leq j < s$.

We therefore attempt first to evaluate the first two moments of the random vector $\underline{X}(n)$.

Define the multivariate moment generating function

$$\psi_n(\underline{t}) = E\left[\exp \sum_{\ell=0}^{s} t_\ell X_\ell(n)\right] ,$$

where $\underline{t} = (t_0,\ldots,t_\ell)$. From the recurrence (5.1)

$$\exp \sum_{\ell=0}^{s} t_\ell X_\ell(n+1) = \frac{1}{m}\left(X_0(n) + \sum_{\ell=0}^{s} X_\ell(n) e^{t_{\ell-1}-t_\ell} \right)$$

$$\exp \sum_{\ell=0}^{s} t_\ell X_\ell(n) \ .$$

Calling temporarily

$$\alpha_\ell(t) = \begin{cases} \frac{1}{m} & \text{if} \quad \ell = 0 \ , \\ \frac{1}{m} e^{t_{\ell-1}-t_\ell} & \text{if} \quad 0 < \ell \leq s \ , \end{cases}$$

and taking expectation

$$\psi_{n+1}(\underline{t}) = \sum_{\ell=0}^{s} \alpha_\ell(\underline{t}) \frac{\partial}{\partial t_\ell} \psi_n(\underline{t}) \ . \tag{5.4}$$

Now

$$\frac{\partial \psi_{n+1}(\underline{t})}{\partial t_i} = \sum_{\ell=1}^{s} \frac{\partial \alpha_\ell(\underline{t})}{\partial t_i} \frac{\partial \psi_n(\underline{t})}{\partial t_\ell}$$

$$+ \sum_{\ell=0}^{s} \alpha_\ell(t) \frac{\partial^2 \psi_n(\underline{t})}{\partial t_i \partial t_\ell} \ ,$$

whence by setting $\underline{t} = \underline{0}$.

$$E[X_i(n+1)] = \sum_{\ell=1}^{s} \frac{\partial\alpha_\ell(\underline{0})}{\partial t_i} E[X_\ell(n)]$$

$$+ \frac{1}{m} \sum_{\ell=0}^{s} E[X_i(n)X_\ell(n)] .$$

However $\sum_{\ell=0}^{s} E[X_i(n)X_\ell(n)] = E[X_i(n) \sum_{\ell=0}^{s} X_\ell(n)] = m\, E[X_i(n)]$

since $\sum_{\ell=0}^{s} X_\ell(n) = m$.

Further for $0 < \ell \leq s$

$$\frac{\partial\alpha_\ell(\underline{0})}{\partial t_i} = \begin{cases} \frac{-1}{m} & \text{if} \quad i = \ell , \\ \frac{1}{m} & \text{if} \quad i = \ell - 1 , \\ 0 & \text{otherwise} , \end{cases}$$

so that

$$E[X_0(n+1)] = E[X_0(n)] + \frac{1}{m} E[X_1(n)] ,$$

$$E[X_i(n+1)] = (1 - \frac{1}{m})E[X_i(n)] + \frac{1}{m} E[X_{i+1}(n)]$$

for $0 < i < s$, and

$$E[X_s(n+1)] = (1 - \frac{1}{m})E[X_s(n)] .$$

Denoting $\underline{\tilde{X}}(n)$ the column subvector

$$(X_1(n),\ldots,X_s(n))^T$$

and by Q the $s \times s$ matrix

$$Q = [q_{ij}^{(1)}]$$

with
$$q_{ij}^{(1)} = \begin{cases} 1 - \frac{1}{m} & \text{if} \quad i = j \ , \\ \frac{1}{m} & \text{if} \quad j = i + 1 \ , \\ 0 & \text{otherwise} \ , \end{cases}$$

we can write

$$E[\underset{\sim}{X}(n+1)] = QE[\underset{\sim}{X}(n)] \ ,$$

whence

$$E[\underset{\sim}{X}(n)] = Q^n \underset{\sim}{y}$$

with $\underset{\sim}{y} = (y_1,\ldots,y_s)^T$.

However, Q^n is an upper triangular matrix with entries

$$q_{ij}^{(n)} = \begin{cases} \frac{1}{m^n}\binom{n}{j-i}(m-1)^{n-j+i} & \text{if} \quad j \geq i \ , \\ 0 & \text{if} \quad j < i \ , \end{cases}$$

as is readily verified by induction. Hence

$$E[X_i(n)] = \frac{1}{m^n} \sum_{j=i}^{s} \binom{n}{j-i} (m-1)^{n-j+i} \; y_j$$

(5.6)

$$= \frac{1}{m^n} \sum_{\ell=0}^{s-i} \binom{n}{\ell} (m-1)^{n-\ell} \; y_{\ell+i} \; ,$$

$i = 1,\dots,s$, is the desired expectation vector.

The expectation of $X_0(n)$ is obtained from (5.2),

$$E[X_0(n)] = m - \frac{1}{m^n} \sum_{i=1}^{s} \sum_{\ell=0}^{s-i} \binom{n}{\ell} (m-1)^{n-\ell} \; y_{\ell+i}$$

$$= m - \frac{1}{m^n} \sum_{\ell=0}^{s-1} \binom{n}{\ell} (m-1)^{n-\ell} \; Z_{\ell+1} \; ,$$

where $Z = y + \dots + y_s$, $\ell = 1,\dots,s$.

The expectation of the random vector $\underline{K}(n)$ is obtained using the linear relation (5.3),

$$E[K_j(n)] = Z_j - \frac{1}{m^n} \sum_{\ell=0}^{s-1} \binom{n}{\ell} (m-1)^{n-\ell} \; Z_{j+\ell+1}$$

for $0 < j \le s$, and

$$E[K_0(n)] = Z_1 - \frac{1}{m^n} \sum_{\ell=0}^{s-1} \binom{n}{\ell} (m-1)^{n-\ell} \; Z_{\ell+1} \; .$$

Recurrence equations for the second moments can be developed similarly. Taking second partial derivatives of (5.4) we get

$$\frac{\partial^2 \psi_{n+1}(\underline{t})}{\partial t_i \partial t_j} = \sum_{\ell=1}^{s} \frac{\partial^2 \alpha_\ell(\underline{t})}{\partial t_i \partial t_j} \frac{\partial \psi_n(\underline{t})}{\partial t_\ell}$$

$$+ \sum_{\ell=1}^{s} \frac{\partial \alpha_\ell(\underline{t})}{\partial t_i} \frac{\partial^2 \psi_n(\underline{t})}{\partial t_j \partial t_\ell}$$

$$+ \sum_{\ell=1}^{s} \frac{\partial \alpha_\ell(\underline{t})}{\partial t_j} \frac{\partial^2 \psi_n(\underline{t})}{\partial t_i \partial t_\ell}$$

$$+ \sum_{\ell=0}^{s} \alpha_\ell(\underline{t}) \frac{\partial^3 \psi_n(\underline{t})}{\partial t_i \partial t_j \partial t_\ell}$$

Again upon setting $\underline{t} = 0$ the last terms becomes just $E[X_i(n)X_j(n)]$ and using (5.5) and

$$\frac{\partial^2 \alpha_\ell(\underline{0})}{\partial t_i \partial t_j} = \begin{cases} \frac{1}{m} & \text{if} \quad i = j = \ell \ \text{ or } \ i = j = \ell - 1 \,, \\ -\frac{1}{m} & \\ 0 & \text{otherwise} \,, \end{cases}$$

where $0 \leq i \leq j \leq s$, $0 < \ell \leq s$, we obtain for $0 < i \leq j \leq s$

$$
\begin{aligned}
E[X_i(n+1)X_j(n+1)] = {} & E[X_i(n)X_j(n)] \\
& - \frac{1}{m} E[X_i^2(n)] - \frac{1}{m} E[X_j^2(n)] \\
& + \frac{1}{m} E[X_i(n)X_{i+1}(n)] + \frac{1}{m} E[X_j(n)X_{j+1}(n)] \\
& + \begin{cases} \frac{1}{m} E[X_i(n)] + \frac{1}{m} E[X_{i+1}(n)] & \text{if} \quad j = i \, , \\ - \frac{1}{m} E[X_j(n)] & \text{if} \quad j = i + 1 \, , \\ 0 & \text{if} \quad j > i + 1 \, , \end{cases}
\end{aligned}
$$

with terms $E[X_s(n)X_{s+1}(n)]$ and $E[X_{s+1}(n)]$ being interpreted as zero.

This again looks better in matrix notation. Denote

$$M_n = [\mu_{ij}(n)] , \ A = [\alpha_{ij}] , \ E_n = [E_{ij}(n)]$$

the $s \times s$ matrices defined by

$$\mu_{ij}(n) = E[X_i(n)X_j(n)] ,$$

$$\alpha_{ij} = \begin{cases} \frac{1}{m} & \text{if} \quad j = i , \\ \frac{1}{m} & \text{if} \quad j = i + 1 , \\ 0 & \text{otherwise} , \end{cases}$$

$$E_{ij}(n) = \begin{cases} \frac{1}{m} E[X_i(n)] + \frac{1}{m} E[X_{i+1}(n)] & \text{if} \quad j = i , \\ -\frac{1}{m} E[X_j(n)] & \text{if} \quad j = i + 1 , \\ -\frac{1}{m} E[X_i(n)] & \text{if} \quad i = j + 1 , \\ 0 & \text{otherwise.} \end{cases}$$

Then for $n = 0,1,\ldots$

(5.7) $$M_{n+1} = M_n + AM_n + M_nA^T + E_n ,$$

with T denoting transposition.

Since the entries of M_0 are just y_iy_j while the entries of E_n are available from (5.6) for all $n = 0,1,\ldots,$ the second moment matrix M_n can be evaluated iteratively.

It is even possible to write a closed form expression since upon iterating (5.7) we obtain

$$(5.8) \qquad M_n = \sum_{\nu=0}^{n} \sum_{\ell=0}^{\nu} \binom{n}{\nu}\binom{\nu}{\ell} A^{\ell} M_0 (A^{\nu-\ell})^T$$

$$+ \sum_{k=0}^{n-1} \sum_{\nu=0}^{k} \sum_{\ell=0}^{\nu} \binom{k}{\nu}\binom{\nu}{\ell} A^{\ell} E_{n-1-k} (A^{\nu-\ell})^T ,$$

$n = 1,2,\ldots$. The ℓ-th power $A^{\ell} = [\alpha_{ij}^{(\ell)}]$ of the matrix A is an upper triangular matrix with entries

$$\alpha_{ij}^{(\ell)} = \begin{cases} \dfrac{(-1)^{j-i}}{m^{\ell}} \dbinom{\ell}{j-i} & \text{if} \quad j \geq i , \\ 0 & \text{if} \quad j < i , \end{cases}$$

so that all the terms on the right-hand side of (5.8) are available.

Having the mean vector and the second moment matrix one can calculate the covariance matrix, from which the covariance matrix of the random vector $\underline{K}(n)$ is obtain by using the linear transformation (5.3). Although having an expression for the mean vector and the covariance matrix is useful one would like to obtain a formula for the probability distribution as we did in the univariate case. Unfortunately, the multivariate case is considerably more complicated and only a partial result is obtained.

Let $0 < j \leq s$ be fixed and denote by $X_j^{(i)}(n)$ the number of cells among the original y_i cells (with ball type i initially) that after n trials contain balls of type j. Clearly

$$X_j(n) = X_j^{(0)}(n) + \ldots + X_j^{(s)}(n) ,$$

where of course $X_j^{(i)}(n) = 0$ for $i < j$.

Let us now condition on the random vector $\underline{N} = (N_0, \ldots N_s)$, where N_i is the number of trials which resulted is a selection of a cell from the original group of y_i cells. Then $\underline{N}$ is multinomial with parameters n and y_i/m, $i = 0, \ldots, s$, and $X_j^{(0)}(n), \ldots, X_j^{(s)}(n)$ are conditionally independent given $\underline{N}$. It follows that

$$\text{(5.9)} \qquad P(X_j(n) = k) = \sum \binom{n}{n_0, \ldots, n_s} \frac{y_0^{n_0} \cdots y_s^{n_s}}{m^n}$$

$$P(X_j^{(0)}(n) = k|\underline{N}) * P(X_j^{(1)}(n) = k|\underline{N}) * \ldots$$

$$* P(X_j^{(s)}(n) = k|\underline{N}) ,$$

where the summation is over all nonnegative integers $n_0, \ldots, n_s$ such that $n_0 + \ldots + n_s = n$ and asterisks denote convolutions.

Now for $i < j$ we have trivially

$$P(X_j^{(i)}(n) = 0|\underline{N}) = 1 ,$$

while for $j \le i \le s$ the conditional distribution $P(X_j^{(i)}(n) = k \mid \underline{N} = \underline{n})$ is identical with the classical occupancy distribution $p_{n_i}^{(y)}(k)$ of finding exactly k cell occupied with exactly $i - j$ balls after distributing n_i balls randomly into $y = y_i$ initially empty cells.

This distribution is most easily derived from the generating function

$$H_j^{(y)}(u,v) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(yu)^n}{n!} v^k p_n^{(y)}(k)$$

by conditioning on the number of balls placed in each cell. (See [JK], p. 116 for details.) It follows that

$$H_j^{(y)}(u,v) = \left(H_j^{(1)}(u,v)\right)^y$$

and since $p_n^{(1)}(1) = 1$ if $n = i-j$ while $p_n^{(1)}(0) = 1$ otherwise we have

$$(5.10) \qquad H_j^{(y)}(u,v) = \left[(v-1)\frac{u^{i-j}}{(i-j)!} + e^u\right]^y .$$

Now for $0 \leq k \leq y$

$$\left.\frac{\partial^k H_j^{(y)}}{\partial v^k}\right|_{v=0} = y^{(k)}\left(\frac{u^{i-j}}{(i-j)!}\right)^k \left|e^u - \frac{u^{i-j}}{(i-j)!}\right|^{y-k}$$

$$= y^{(k)} \sum_{\ell=0}^{y-k} (-1)^\ell \binom{y-k}{\ell} \left(\frac{u^{i-j}}{(i-j)!}\right)^{k+\ell} e^{(y-k-\ell)u}$$

$$= y^{(k)} \sum_{\nu=0}^{\infty} \sum_{\ell=0}^{y-k} (-1)^\ell \binom{y-k}{\ell} \frac{(y-k+\ell)^\nu}{\nu!((i-j)!)^{k+\ell}} u^{(i-j)(k+\ell)+\nu} ,$$

so that $P_n^{(y)}(k) = \frac{n!}{y^n} \times$ coefficient at u^n .

Thus with $y = y_i$ and $n = n_i$ we have

$$P_{n_i}^{(y_i)}(k) = \frac{y_i^{(k)} n_i!}{y_i^{n_i}} \sum_{\ell=0}^{y_i-k} (-1)^\ell \binom{y_i-k}{\ell}$$

$$\times \frac{(y-k+\ell)^{n_i - (i-j)(k+\ell)}}{((i-j)!)^{k+\ell}(n_i - (i-j)(k+\ell))!} .$$

Upon substitution into (5.9) we obtain the desired expression for the probabilities (for $0 < j \leq s$)

$$P(X_j(n) = k) = \frac{n!}{m^n} \sum \sum \frac{(y_0 + \ldots + y_{j-1})^{n - (n_j + \ldots + n_s)}}{(n - (n_j + \ldots + n_s))!}$$

$$\sum_{\ell_i=0}^{y_i-k_i} \frac{(-1)^{\ell_i}}{\ell_i!(y_i-k_i-\ell_i)!} \frac{(y_i-k_i+\ell_i)^{n_i-(i-j)(k_i+\ell_i)}}{((i-j)!)^{k_i+\ell_i}(n_i-(i-j)(k_i+\ell_i))!} ,$$

where the first two sums are, respectively, over all nonnegative integers $n_j,\ldots,n_s$ and $k_j,\ldots,k_s$ such that

$$n_j + \ldots + n_s \leq n ,$$

and

$$k_j + \ldots + k_s = k .$$

Unfortunately, for larger values of s, k, or n the number of terms involved in the indicated sums is rather prohibitive. The problem of developing a manageable computational algorithm to evaluate the probabilities $P(X_j(n) = k)$ is left for future investigation.

It should be pointed out, however, that the generating function

$$G_j(u,v) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(mu)^n}{n!} v^k P(X_j(n) = k)$$

is just a product of the generating functions (5.10), specifically

$$G_j(u,v) = e^{(y_0+\ldots+y_{j-1})u} H_j^{(y_j)}(u,v)\ldots H_j^{(y_s)}(u,v) .$$

Thus, if an algorithm can be found to expand the right-hand side of this expression into a power series in u and v the probabilities could be determined directly from this expansion.

As an example of this approach consider the case when the parameters $y_1,\ldots,y_s$ are themselves random, having a multinomial distribution with parameters s, $b = y_1+\ldots+y_s$, and $p_i = 1/s$, $i = 1,\ldots,s$, respectively. In other words assume that each ball type $0 < j \leq s$ was selected independently and at random among all nonzero ball types $1,\ldots,s$. Their using the multinomial theorem we see that

$$G_1(u,v) = e^{y_0 u}\left[be^u + (v-1)\sum_{i=1}^{s}\frac{u^{i-1}}{(i-1)!}\right]^b .$$

Then taking k-th partial derivative with respect to v and setting $v = 0$ we get

$$b^{(k)}\sum_{\ell=0}^{b-k}(-1)^{\ell}\left(\sum_{i=0}^{s-1}\frac{u^i}{i!}\right)^{k+\ell} b^{(b-k-\ell)}\, e^{(m-k-\ell)u} .$$

After expanding the exponential function it is seen that the coefficients of powers of u will be expressible as ordinary double sums, involving coefficients of powers of the polynomial

$$\sum_{i=0}^{s-1}\frac{u^i}{i!} .$$

The latter could possibly be precomputed and thus there is a hope for a reasonably efficient algorithm.

References

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